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Axially symmetric problems for a porous elastic solid

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Abstract

This paper is concerned with the linear theory of elastic materials with voids. The Dirichlet and Neumann problems for a half-space are studied by using the technique of integral transforms. The case of a concentrated body load is investigated in detail.

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1. Introduction

The theory of materials with voids was initiated by Nunziato and Cowin (1979). In recent years this theory has been a subject of intensive study. The intended application of the theory are to geological materials and to manufactured porous materials. In this theory, the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom. The linear theory of elastic material with voids has been established by Cowin and Nunziato (1983). Various applications of the theory were presented by Cowin and Puri (1983), Cowin (1983), Chandrasekharaiah (1989), Scarpetta (1990) and Ciarletta and Ieşan (1993).

In this paper we consider the equilibrium theory of an elastic material with voids that occupies a half-space and is subjected to an axially symmetric deformation. First, we employ the technique of integral transforms to obtain a general solution of the Lamé equations for an arbitrary system of loads. Then, we study the case of a half-space with a fixed boundary and subjected to a concentrated extrinsic equilibrated body force. The stresses have been evaluated at the boundary and the displacement field and the volume fraction field have been determined in the interior of the body. The problem of a half-space with a stress-free boundary is also investigated.

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2. Basic equations

Throughout this section B is a regular region of the three-dimensional Euclidean space. We let ∂B denote the boundary of B, and designate by $\mathbf{n}$ the outward unit normal of ∂B. We assume that the region B is occupied by a linearly elastic material with voids. The body is referred to a system of rectangular Cartesian axes Ox_i. Throughout this paper, Latin indices have the range 1–3. Let $\mathbf{u}$ be the displacement field over B. The linear strain measure e_{ij} is given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.1}$$

We use subscripts preceded by a comma for partial differentiation with respect to the corresponding coordinate.

Let t_{ij} be the stress tensor and let h_i be the equilibrated stress vector. The components of surface traction t_i and the equilibrated stress h at a regular point of ∂B are given by

$$t_i = t_{ji}n_j, \qquad h = h_i n_i, \tag{2.2}$$

respectively.

The equilibrium equations are

$$t_{ji,j} + f_i = 0, \qquad h_{i,i} + g + l = 0, \tag{2.3}$$

where f_i are the components of body force, g is the intrinsic equilibrated body force, and l is the extrinsic equilibrated body force.

In the case of centrosymmetric isotropic material the constitutive equations are

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \beta\varphi\delta_{ij}, \\ h_i &= \alpha\varphi_{,i}, \\ g &= -\beta e_{rr} - \zeta\varphi, \end{aligned} \tag{2.4}$$

where φ is the volume fraction function, δ_{ij} is Kronecker's delta, and λ, μ, β, α and ζ are constitutive coefficients. We restrict our attention to homogeneous materials so that the constitutive coefficients are constants. We assume that the internal energy density is a positive definite form. This assumption implies that (Cowin and Nunziato, 1983)

$$\mu > 0, \quad \alpha > 0, \quad \zeta > 0, \quad 2\mu + 3\lambda > 0, \quad (2\mu + 3\lambda)\zeta > 3b^2. \tag{2.5}$$

3. Axially symmetric problems

We assume that the region B from here on refers to the half-space $x_3 > 0$. In what follows we are interested in axially symmetric problems with the displacement field and the volume fraction being specified in cylindrical coordinates (r, ϑ, z) as follows

$$u_r = u(r,z), \quad u_\vartheta = 0, \quad u_z = w(r,z), \quad \varphi = \varphi(r,z), \quad (r,z) \in \Im. \tag{3.1}$$

The geometrical equations become

$$e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\vartheta\vartheta} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad e_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \quad e_{r\vartheta} = 0, \quad e_{\vartheta z} = 0. \tag{3.2}$$

The constitutive equations reduce to

$$
\begin{aligned}
&t_{rr} = \lambda e + 2\mu e_{rr} + \beta\varphi, \qquad t_{\vartheta\vartheta} = \lambda e + 2\mu r^{-1}u + \beta\varphi,\\
&t_{zz} = \lambda e + 2\mu e_{zz} + \beta\varphi, \qquad t_{r\vartheta} = t_{\vartheta z} = 0,\\
&t_{rz} = 2\mu e_{rz}, \qquad h_r = \alpha\frac{\partial\varphi}{\partial r}, \qquad h_\vartheta = 0,\\
&h_z = \alpha\frac{\partial\varphi}{\partial z}, \qquad g = -\beta e - \zeta\varphi,
\end{aligned}
\tag{3.3}
$$

where

$$
e = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}. \tag{3.4}
$$

In the case of axisymmetry the equilibrium equations take the form

$$
\begin{aligned}
&\frac{\partial t_{rr}}{\partial r} + \frac{\partial t_{rz}}{\partial z} + \frac{1}{r}(t_{rr} - t_{\vartheta\vartheta}) + f_r = 0,\\
&\frac{\partial t_{rz}}{\partial r} + \frac{\partial t_{zz}}{\partial z} + \frac{1}{r}t_{rz} + f_z = 0,\\
&\frac{1}{r}\frac{\partial}{\partial r}(rh_r) + \frac{\partial h_z}{\partial z} + g + l = 0.
\end{aligned}
\tag{3.5}
$$

Eqs. (3.5) can be expressed in terms of u, w and φ

$$
\begin{aligned}
&\mu\left(\Delta - \frac{1}{r^2}\right)u + (\lambda+\mu)\frac{\partial e}{\partial r} + \beta\frac{\partial\varphi}{\partial r} + f_r = 0,\\
&\mu\Delta w + (\lambda+\mu)\frac{\partial e}{\partial z} + \beta\frac{\partial\varphi}{\partial z} + f_z = 0,\\
&\alpha\Delta\varphi - \beta e - \zeta\varphi + l = 0,
\end{aligned}
\tag{3.6}
$$

where

$$
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3.7}
$$

We assume that all stresses, displacements and volume fraction vanish at infinity.

We introduce the following Hankel-transforms (Sneddon, 1972)

$$
\begin{aligned}
&U(\xi,z) = \int_0^\infty ru(r,z)J_1(\xi r)\,\mathrm{d}r,\\
&W(\xi,z) = \int_0^\infty rw(r,z)J_0(\xi r)\,\mathrm{d}r,\\
&\Phi(\xi,z) = \int_0^\infty r\varphi(r,z)J_0(\xi r)\,\mathrm{d}r,
\end{aligned}
\tag{3.8}
$$

where $J_n(z)$ is the Bessel function of the first kind and nth order. Applying Hankel transformation to Eqs. (3.6), we obtain

$$
\begin{aligned}
&[\mu D^2 - (\lambda+2\mu)\xi^2]U - (\lambda+\mu)\xi DW - \beta\xi\Phi + F = 0,\\
&(\lambda+\mu)\xi DU + [(\lambda+2\mu)D^2 - \mu\xi^2]W + \beta D\Phi + G = 0,\\
&[\alpha(D^2 - \xi^2) - \zeta]\Phi - \beta\xi U - \beta DW + L = 0,
\end{aligned}
\tag{3.9}
$$

where we have used the notations

$$D = \frac{\mathrm{d}}{\mathrm{d}z}, \qquad F(\xi, z) = \int_0^\infty r f_r(r, z) J_1(\xi r)\,\mathrm{d}r,$$
$$G(\xi, z) = \int_0^\infty r f_z(r, z) J_0(\xi r)\,\mathrm{d}r, \qquad L(\xi, z) = \int_0^\infty r l(r, z) J_0(\xi r)\,\mathrm{d}r. \tag{3.10}$$

In what follows we designate by $\overline{g}$ the Laplace transform of the function g with respect to z, i.e.,

$$\overline{g}(\xi, p) = \int_0^\infty g(\xi, z)\mathrm{e}^{-pz}\,\mathrm{d}z, \quad \mathrm{Re}\, p > 0.$$

From (3.9) we obtain the equations

$$\begin{aligned}
&[\mu p^2 - (\lambda + 2\mu)\xi^2]\overline{U} - (\lambda + \mu)\xi p \overline{W} - \beta\xi\overline{\Phi} = R,\\
&(\lambda + \mu)\xi p \overline{U} + [(\lambda + 2\mu)p^2 - \mu\xi^2]\overline{W} + \beta p \overline{\Phi} = S,\\
&[\alpha(p^2 - \xi^2) - \zeta]\overline{\Phi} - \beta\xi\overline{U} - \beta p \overline{W} = T,
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
&R = \mu(U^{(1)} + pU^{(0)}) - (\lambda + \mu)\xi W^{(0)} - \overline{F},\\
&S = (\lambda + \mu)\xi U^{(0)} + (\lambda + 2\mu)(W^{(1)} + pW^{(0)}) - \overline{G},\\
&T = \alpha(\Phi^{(1)} + p\Phi^{(0)}) - \beta W^{(0)} - \overline{L},\\
&U^{(1)}(\xi) = (DU)(\xi, 0), \qquad U^{(0)}(\xi) = U(\xi, 0),\\
&W^{(1)}(\xi) = (DW)(\xi, 0), \qquad W^{(0)}(\xi) = W(\xi, 0),\\
&\Phi^{(1)}(\xi) = (D\Phi)(\xi, 0), \qquad \Phi^{(0)}(\xi) = \Phi(\xi, 0).
\end{aligned} \tag{3.12}$$

We introduce the notations

$$\begin{aligned}
&q^2 = \frac{1}{\alpha(\lambda + 2\mu)}[\zeta(\lambda + 2\mu) - \beta^2], \qquad \tau = \sqrt{\xi^2 + q^2},\\
&\Lambda = \alpha(p^2 - \xi^2)^2(p^2 - \tau^2), \qquad a = \mu\alpha(\lambda + 2\mu),\\
&\Gamma_1 = RM + SN + TQ, \qquad \Gamma_2 = RM^* + SN^* + TQ^*,\\
&\Gamma_3 = \mu(p^2 - \xi^2)[\beta\xi R + \beta p S + (\lambda + 2\mu)(p^2 - \xi^2)T],\\
&M = \alpha\mu(p^2 - \xi^2)^2 + (p^2 - \xi^2)[\alpha(\lambda + \mu)p^2 - \zeta\mu] + p^2[\beta^2 - \zeta(\lambda + \mu)],\\
&N = p\xi[\alpha(\lambda + \mu)(p^2 - \xi^2) + \beta^2 - \zeta(\lambda + \mu)],\\
&Q = \beta\mu\xi(p^2 - \xi^2), \qquad M^* = -p\xi\{(\lambda + \mu)[\alpha(p^2 - \xi^2) - \zeta] + \beta^2\},\\
&N^* = [\mu(p^2 - \xi^2) - (\lambda + \mu)\xi^2][\alpha(p^2 - \xi^2) - \zeta] - \beta^2\xi^2,\\
&Q^* = -\beta\mu p(p^2 - \xi^2).
\end{aligned} \tag{3.13}$$

It follows from (3.11) that for Λ different from zero we have

$$(\overline{U}, \overline{W}, \overline{\Phi}) = \frac{1}{\Lambda}(\Gamma_1, \Gamma_2, \Gamma_3). \tag{3.14}$$

The function $\overline{U}$, $\overline{W}$ and $\overline{\Phi}$ must have no singularities in the right half-plane of the variable p. This fact implies the conditions

$$\Gamma_k(\xi,\xi)=0, \qquad \frac{\partial \Gamma_k}{\partial p}(\xi,\xi)=0, \qquad \Gamma_k(\xi,\tau)=0. \tag{3.15}$$

The conditions (3.15) reduce to

$$\begin{aligned}
&R(\xi,\xi)+S(\xi,\xi)=0,\\
&\gamma_1 R(\xi,\xi)+\gamma_2 S(\xi,\xi)+\gamma_3 T(\xi,\xi)+\xi\Sigma(\xi)=0,\\
&k[\xi R(\xi,\tau)+\tau S(\xi,\tau)]+q^2 T(\xi,\tau)=0,
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
&\rho\gamma_1=2\alpha[(\lambda+\mu)\xi^2-(\lambda+2\mu)q^2], \qquad \rho\gamma_2=(\lambda+\mu)(2\alpha\xi^2-\zeta)+\beta^2,\\
&\rho\gamma_3=2\beta\mu\xi, \rho=\beta^2-\zeta(\lambda+\mu), \qquad k=\frac{\beta}{\lambda+2\mu},\\
&\Sigma(\xi)=\mu U^{(0)}(\xi)+(\lambda+2\mu)W^{(0)}(\xi)-\frac{\partial\overline{F}}{\partial p}(\xi,\xi)-\frac{\partial\overline{G}}{\partial p}(\xi,\xi).
\end{aligned} \tag{3.17}$$

It is easy to see that the relations (2.5) imply that ρ is different from zero.

In view of (3.12), the relations (3.16) take the form

$$\begin{aligned}
&\mu(U^{(1)}+\xi U^{(0)})+(\lambda+\mu)\xi(U^{(0)}-W^{(0)})+(\lambda+2\mu)(W^{(1)}+\xi W^{(0)})=(\overline{F}+\overline{G})(\xi,\xi),\\
&\gamma_1\left[\mu(U^{(1)}+\xi U^{(0)})-(\lambda+\mu)\xi W^{(0)}\right]+\gamma_2\left[(\lambda+\mu)\xi U^{(0)}+(\lambda+2\mu)(W^{(1)}+\xi W^{(0)})\right]\\
&\quad+\gamma_3\left[\alpha(\Phi^{(1)}+\xi\Phi^{(0)})-\beta W^{(0)}\right]+\xi\left[\mu U^{(0)}+(\lambda+2\mu)W^{(0)}\right]\\
&\quad=\gamma_1\overline{F}(\xi,\xi)+\gamma_2\overline{G}(\xi,\xi)+\gamma_3\overline{L}(\xi,\xi)+\xi\left(\frac{\partial\overline{F}}{\partial p}+\frac{\partial\overline{G}}{\partial p}\right)(\xi,\xi),\\
&k\left\{\xi\left[\mu(U^{(1)}+\tau U^{(0)})-(\lambda+\mu)\xi W^{(0)}\right]+\tau\left[(\lambda+\mu)\xi U^{(0)}+(\lambda+2\mu)(W^{(1)}+\tau W^{(0)})\right]\right\}\\
&\quad+q^2\left[\alpha(\Phi^{(1)}+\tau\Phi^{(0)})-\beta W^{(0)}\right]=k\xi\overline{F}(\xi,\tau)+k\tau\overline{G}(\xi,\tau)+q^2\overline{L}(\xi,\tau).
\end{aligned} \tag{3.18}$$

In what follows we consider two cases: (i) a half-space with a fixed boundary; (ii) a half-space with a stress-free boundary. The method of integral transforms has been used to study axisymmetric problems in various theories of continua (see, e.g. Nowacki, 1971; Sneddon, 1962; Khan and Dhaliwal, 1977).

4. The Dirichlet problem—concentrated body loads

In this section we consider that the boundary $z=0$ is fixed and consider the boundary conditions

$$u=0, \quad w=0, \quad \varphi=0 \quad \text{at } z=0. \tag{4.1}$$

In this case we have $U^{(0)}=0$, $W^{(0)}=0$, $\Phi^{(0)}=0$ and the conditions (3.18) reduce to

$$\begin{aligned}
&\mu U^{(1)}+(\lambda+2\mu)W^{(1)}=N_1,\\
&\gamma_1\mu U^{(1)}+\gamma_2(\lambda+2\mu)W^{(1)}+\gamma_3\alpha\Phi^{(1)}=N_2,\\
&k\xi\mu U^{(1)}+k\tau(\lambda+2\mu)W^{(1)}+q^2\alpha\Phi^{(1)}=N_3,
\end{aligned} \tag{4.2}$$

where

$$
\begin{aligned}
&N_1(\xi) = \overline{F}(\xi,\xi) + \overline{G}(\xi,\xi),\\
&N_2(\xi) = \gamma_1\overline{F}(\xi,\xi) + \gamma_2\overline{G}(\xi,\xi) + \gamma_3\overline{L}(\xi,\xi) + \xi\left(\frac{\partial\overline{F}}{\partial p} + \frac{\partial\overline{G}}{\partial p}\right)(\xi,\xi),\\
&N_3(\xi) = k\xi\overline{F}(\xi,\tau) + k\tau\overline{G}(\xi,\tau) + q^2\overline{L}(\xi,\tau).
\end{aligned}
\tag{4.3}
$$

The solution of the system (4.2) is given by

$$
\mu U^{(1)} = \frac{1}{d}d_1, \quad (\lambda+2\mu)W^{(1)} = \frac{1}{d}d_2, \quad \alpha\Phi^{(1)} = \frac{1}{d}d_3,
\tag{4.4}
$$

where

$$
\begin{aligned}
&d = (\gamma_2-\gamma_1)q^2 - k\gamma_3(\tau-\xi), \qquad d_1 = N_1(q^2\gamma_2 - k\tau\gamma_3) - N_2q^2 + \gamma_3N_3,\\
&d_2 = q^2N_2 - \gamma_3N_3 - N_1(q^2\gamma_1 - k\gamma_3\xi),\\
&d_3 = (\gamma_2-\gamma_1)(N_3 - k\xi N_1) - k(\tau-\xi)(N_2 - \gamma_1N_1),\\
&\rho(\gamma_2-\gamma_1) = (\lambda+3\mu)\zeta - \beta^2.
\end{aligned}
\tag{4.5}
$$

We introduce the Hankel-transforms

$$
\begin{aligned}
&T_{zz}(\xi,z) = \int_0^\infty rt_{zz}(r,z)J_0(\xi r)\,\mathrm{d}r,\\
&T_{zr}(\xi,z) = \int_0^\infty rt_{zr}(r,z)J_1(\xi r)\,\mathrm{d}r,\\
&\Pi_z(\xi,z) = \int_0^\infty rh_z(r,z)J_0(\xi r)\,\mathrm{d}r.
\end{aligned}
\tag{4.6}
$$

It follows from (3.3) that

$$
\begin{aligned}
&T_{zz} = (\lambda+2\mu)DW + \lambda\xi U + \beta\Phi,\\
&T_{zr} = \mu(DU - \xi W), \qquad \Pi_z = \alpha D\Phi.
\end{aligned}
\tag{4.7}
$$

In view of (4.1), (4.4) and (4.7) we obtain

$$
\begin{aligned}
&t_{zz}(r,0) = H_0[Z_2(\xi);\xi\to r],\\
&t_{zr}(r,0) = H_1[Z_1(\xi);\xi\to r],\\
&h_z(r,0) = H_0[Z_3(\xi);\xi\to r],
\end{aligned}
\tag{4.8}
$$

where $Z_k = d_k/d$ and

$$
H_n[g(\xi);\xi\to r] = \int_0^\infty \xi g(\xi)J_n(\xi r)\,\mathrm{d}\xi.
\tag{4.9}
$$

Similarly, from (3.3) and (4.4) we find that

$$
\begin{aligned}
&t_{rr}(r,0) = t_{\vartheta\vartheta}(r,0) = \frac{\lambda}{\lambda+2\mu}t_{zz}(r,0),\\
&h_r(r,0) = 0, \qquad g(r,0) = -\frac{\beta}{\lambda+2\mu}t_{zz}(r,0).
\end{aligned}
\tag{4.10}
$$

Now, we consider the case of a concentrated extrinsic equilibrated body force. We assume that

$$f_r = 0, \qquad f_z = 0, \qquad l = -\frac{P}{2\pi r}\delta(r)\delta(z-h), \quad h > 0, \tag{4.11}$$

where P is a given constant and δ is the Dirac measure. Thus, we obtain

$$\overline{F} = 0, \qquad \overline{G} = 0, \qquad \overline{L} = -\frac{P}{2\pi}\mathrm{e}^{-ph}. \tag{4.12}$$

In this case, from (4.3) we have

$$N_1 = 0, \qquad N_2 = \gamma_3\overline{L}(\xi,\xi), \qquad N_3 = q^2\overline{L}(\xi,\tau). \tag{4.13}$$

By (4.8) we find the following stresses at the boundary

$$\begin{aligned}
t_{zz}(r,0) &= -\frac{P}{\pi\rho}\beta\mu q^2\int_0^\infty \frac{1}{\mathrm{d}(\xi)}\xi^2(\mathrm{e}^{-\xi h} - \mathrm{e}^{-\tau h})J_0(\xi r)\,\mathrm{d}\xi,\\
t_{zr}(r,0) &= -\frac{P}{\pi\rho}\beta\mu q^2\int_0^\infty \frac{1}{\mathrm{d}(\xi)}\xi^2(\mathrm{e}^{-\tau h} - \mathrm{e}^{-\xi h})J_1(\xi r)\,\mathrm{d}\xi,\\
h_z(r,0) &= -\frac{P}{2\pi\rho}\int_0^\infty \frac{1}{\mathrm{d}(\xi)}[(\gamma_2-\gamma_1)q^2\rho\mathrm{e}^{-\tau h} - 2k(\tau-\xi)\beta\mu\xi\mathrm{e}^{-\xi h}]\xi J_0(\xi r)\,\mathrm{d}\xi.
\end{aligned} \tag{4.14}$$

From (4.10) we can obtain t_{rr}, $t_{\vartheta\vartheta}$ and g for $z = 0$.

Let us study now the effects inside the body. It follows from (3.13) and (3.14) that

$$(U, W, \Phi) = \frac{1}{2\pi\mathrm{i}a}\int_C \frac{\mathrm{e}^{zp}}{(p^2-\tau^2)(p^2-\xi^2)^2}(\Gamma_1,\Gamma_2,\Gamma_3)(\xi,p)\,\mathrm{d}p, \tag{4.15}$$

where C is the Bromwich path of integration and $\mathrm{i} = \sqrt{-1}$. In the case of the body loads (4.11) the functions Γ_k have the form

$$\Gamma_k = A_k\mathrm{e}^{-ph} + B_k, \tag{4.16}$$

where A_k and B_k are polynomials in p with coefficients that depend on ξ. In the right-half of p-plane $\overline{U}$, $\overline{W}$ and $\overline{\Phi}$, as functions of p, have no poles. The integrands in (4.15) have one double pole at $p = -\xi$ and a single pole at $p = -\tau$. By using the method of residues we obtain, for $z > h$, the following expressions for U, W and Φ

$$\begin{aligned}
4aq^4\xi^4(U, W, \Phi) = {} & \left[\xi(2\xi^2 - q^2 - \xi q^2 z)(\Gamma_1,\Gamma_2,\Gamma_3)(\xi,-\xi) - q^2\xi^2\frac{\partial}{\partial p}(\Gamma_1,\Gamma_2,\Gamma_3)(\xi,-\xi)\right]\mathrm{e}^{-\xi z}\\
& - \frac{2}{\tau}\xi^4(\Gamma_1,\Gamma_2,\Gamma_3)(\xi,-\tau)\mathrm{e}^{-\tau z}.
\end{aligned} \tag{4.17}$$

From (3.12), (4.1), (4.4), (4.5) and (4.13) we find that

$$\begin{aligned} &R = Z_1, \quad S = Z_2, \quad T = Z_3 - \overline{L}, \\ &d_1 = -d_2 = -\frac{P}{\pi\rho}\beta\mu\xi q^2(\mathrm{e}^{-\tau h} - \mathrm{e}^{-\xi h}), \\ &d_3 = -\frac{P}{2\pi\rho}\left[(\gamma_2 - \gamma_1)\rho q^2\mathrm{e}^{-\tau h} - 2k(\tau - \xi)\beta\mu\xi\mathrm{e}^{-\xi h}\right]. \end{aligned} \tag{4.18}$$

To obtain the functions U, W and Φ from (4.17) we note that

$$\begin{aligned} &\Gamma_1(\xi, -\xi) = \Gamma_2(\xi, -\xi) = -2m^2\xi^2 Z_1, \qquad \Gamma_3(\xi, -\xi) = 0, \\ &m^2 = (\lambda + \mu)\zeta - \beta^2, \quad \Gamma_1(\xi, -\tau) = \xi\Pi, \quad \Gamma_2(\xi, -\tau) = \tau\Pi, \\ &\Pi = k\beta\mu(\xi + \tau)Z_1 + \beta\mu q^2\left(Z_3 + \frac{1}{2\pi}P\mathrm{e}^{\tau h}\right), \\ &\Gamma_3(\xi, -\tau) = \mu q^2\left[\beta(\xi + \tau)Z_1 + (\lambda + 2\mu)q^2\left(Z_3 + \frac{1}{2\pi}P\mathrm{e}^{\tau h}\right)\right], \\ &\frac{\partial\Gamma_1}{\partial p}(\xi, -\xi) = -\xi\left[4\alpha(\lambda + \mu)\xi^2 - 2\alpha(\lambda + 2\mu)q^2 + m^2\right]Z_1 - 2\beta\mu\xi^2\left(Z_3 + \frac{1}{2\pi}P\mathrm{e}^{\xi h}\right), \\ &\frac{\partial\Gamma_2}{\partial p}(\xi, -\xi) = -\xi\left[4\alpha(\lambda + \mu)\xi^2 - m^2 + 2\mu\zeta\right]Z_1 - 2\beta\mu\xi^2\left(Z_3 + \frac{1}{2\pi}P\mathrm{e}^{\xi h}\right), \\ &\frac{\partial\Gamma_3}{\partial p}(\xi, -\xi) = -4\beta\mu\xi^2 Z_1. \end{aligned} \tag{4.19}$$

In the case $z < h$, the functions A_k from (4.16) have no contribution in the expressions of U, W and Φ, and we obtain

$$\begin{aligned} 4aq^4\xi^4(U, W, \Phi) = &-\frac{2}{\tau}\xi^4[(B_1, B_2, B_3)(\xi, -\tau)\mathrm{e}^{-\tau z} - (B_1, B_2, B_3)(\xi, \tau)\mathrm{e}^{\tau z}] \\ &+ \left[\xi(2\xi^2 - q^2 - \xi z q^2)(B_1, B_2, B_3)(\xi, -\xi) - q^2\xi^2\frac{\partial}{\partial p}(B_1, B_2, B_3)(\xi, -\xi)\right]\mathrm{e}^{-\xi z} \\ &+ \left[\xi(q^2 - 2\xi^2 - \xi q^2 z)(B_1, B_2, B_3)(\xi, \xi) - q^2\xi^2\frac{\partial}{\partial p}(B_1, B_2, B_3)(\xi, \xi)\right]\mathrm{e}^{\xi z}, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} &B_1 = \frac{1}{d}[(M - N)d_1 + Qd_3], \qquad B_2 = \frac{1}{d}[(M^* - N^*)d_1 + Q^* d_3], \\ &B_3 = \frac{1}{d}\left[\beta(\xi - p)d_1 + (\lambda + 2\mu)(p^2 - \xi^2)d_3\right]\mu(p^2 - \xi^2). \end{aligned} \tag{4.21}$$

We note that

$$
\begin{aligned}
&M(\xi,\tau)=M(\xi,-\tau)=k\mu\beta\xi^2, \qquad N(\xi,\tau)=-N(\xi,-\tau)=k\beta\mu\tau\xi,\\
&Q(\xi,\tau)=Q(\xi,-\tau)=\beta\mu q^2\tau, \qquad M^*(\xi,\tau)=-M^*(\xi,-\tau)=-k\beta\mu\tau\xi,\\
&N^*(\xi,\tau)=N^*(\xi,-\tau)=-k\beta\mu\tau^2, \qquad Q^*(\xi,\tau)=-Q^*(\xi,-\tau)=-\beta\mu\tau q^2,\\
&M(\xi,\xi)=M(\xi,-\xi)=-m^2\xi^2, \qquad N(\xi,\xi)=-N(\xi,-\xi)=-m^2\xi^2,\\
&Q(\xi,\xi)=Q(\xi,-\xi)=0, \qquad M^*(\xi,\xi)=-M^*(\xi,-\xi)=m^2\xi^2,\\
&N^*(\xi,\xi)=N^*(\xi,-\xi)=m^2\xi^2, \qquad Q^*(\xi,\xi)=-Q^*(\xi,-\xi)=0,\\
&B_1(\xi,-\tau)=\frac{1}{d}\beta\mu\xi y_1, \qquad B_2(\xi,-\tau)=\frac{1}{d}\beta\mu\tau y_1,\\
&B_1(\xi,\tau)=\frac{1}{d}\beta\mu\xi y_2, \qquad B_2(\xi,\tau)=-\frac{1}{d}\beta\mu\tau y_2,\\
&y_1=d_1k(\tau+\xi)+q^2d_3, \qquad y_2=d_1k(\xi-\tau)+q^2d_3,\\
&B_3(\xi,-\tau)=\frac{1}{d}\mu q^2y_3(\xi,\tau), \qquad B_3(\xi,\tau)=\frac{1}{d}\mu q^2y_3(\xi,-\tau),\\
&y_3(\xi,\tau)=\beta(\xi+\tau)d_1+(\lambda+2\mu)q^2d_3,\\
&B_1(\xi,-\xi)=B_2(\xi,-\xi)=-\frac{2}{d}d_1m^2\xi^2,\\
&B_3(\xi,-\xi)=B_3(\xi,\xi)=0, \qquad B_1(\xi,\xi)=B_2(\xi,\xi)=0
\end{aligned}
\tag{4.22}
$$

and

$$
\begin{aligned}
&\frac{\partial B_1}{\partial p}(\xi,\xi)=-\frac{\partial B_2}{\partial p}(\xi,\xi)=\frac{1}{d}\xi E_1, \quad \frac{\partial B_1}{\partial p}(\xi,-\xi)=\frac{1}{d}\xi E_2,\\
&\frac{\partial B_2}{\partial p}(\xi,-\xi)=-\frac{1}{d}\xi E_3,\\
&\frac{\partial B_3}{\partial p}(\xi,\xi)=0, \quad \frac{\partial B_3}{\partial p}(\xi,-\xi)=-\frac{4}{d}\mu\xi^2\beta d_1,\\
&E_1=d_1[\beta^2-\zeta(\lambda+3\mu)]+2\beta\mu\xi d_3,\\
&E_2=d_1[\zeta(3\lambda+5\mu)-3\beta^2-4\alpha(\lambda+\mu)\xi^2]-2\beta\mu\xi d_3,\\
&E_3=d_1[4\alpha(\lambda+\mu)\xi^2+\beta^2-\zeta(\lambda-\mu)]+2\beta\mu\xi d_3.
\end{aligned}
\tag{4.23}
$$

With the help of (4.21)–(4.23), we obtain from (4.20) the following forms for U, W and Φ,

$$
\begin{aligned}
&4aq^4\xi^4U=\frac{2}{d\tau}\xi^5\beta\mu(y_2e^{\tau z}-y_1e^{-\tau z})-\frac{1}{d}\left[2d_1m^2(2\xi^2-q^2-\xi q^2z)+q^2E_2\right]\xi^3e^{-\xi z}-\frac{1}{d}E_1q^2\xi^3e^{\xi z},\\
&4aq^4\xi^4W=-\frac{2}{d}\xi^4\beta\mu(y_1e^{-\tau z}+y_2e^{\tau z})-\frac{1}{d}\left[2d_1m^2(2\xi^2-q^2-\xi q^2z)-q^2E_3\right]\xi^3e^{-\xi z}+\frac{1}{d}E_1q^2\xi^3e^{\xi z},\\
&4aq^4\xi^4\Phi=\frac{2}{d\tau}\xi^4\mu q^2[y_3(\xi,-\tau)e^{\tau z}-y_3(\xi,\tau)e^{-\tau z}]+\frac{4}{d}\xi^4q^2\mu\beta d_1e^{-\xi z}.
\end{aligned}
\tag{4.24}
$$

Taking the inverse Hankel-transformations we find that, for $z < h$, the functions u, v and φ are given by

$$4aq^4u(r,z) = H_1\left\{\frac{2}{\mathrm{d}\tau}\xi\beta\mu(y_2\mathrm{e}^{\tau z} - y_1\mathrm{e}^{-\tau z}) - \frac{1}{\mathrm{d}\xi}\left[2d_1m^2(2\xi^2 - q^2 - \xi q^2 z) + q^2E_2\right]\mathrm{e}^{-\xi z} - \frac{1}{\mathrm{d}\xi}E_1q^2\mathrm{e}^{\xi z};\ \xi \to r\right\},$$
$$4aq^4w(r,z) = H_0\left\{\frac{1}{\mathrm{d}\xi}E_1q^2\mathrm{e}^{\xi z} - \frac{1}{\mathrm{d}\xi}\left[2d_1m^2(2\xi^2 - q^2 - \xi q^2 z) - q^2E_3\right]\mathrm{e}^{-\xi z} - \frac{2}{d}\beta\mu(y_1\mathrm{e}^{-\tau z} + y_2\mathrm{e}^{\tau z});\ \xi \to r\right\},$$
$$4aq^4\varphi(r,z) = H_0\left\{\frac{4}{d}q^2\mu\beta d_1\mathrm{e}^{-\xi z} + \frac{2}{\tau d}q^2\mu[y_3(\xi,-\tau)\mathrm{e}^{\tau z} - y_3(\xi,\tau)\mathrm{e}^{-\tau z}];\ \xi \to r\right\}. \tag{4.25}$$

It is easy to see that the functions u, w and φ given by (4.25) satisfy the boundary conditions (4.1).

We introduce the notations

$$Y_1(\xi) = \frac{1}{\xi}\left[4\alpha q^2(\lambda+\mu)\xi^2 - 2\alpha(\lambda+2\mu)q^4 + 3m^2q^2 - 2m^2(2\xi^2 - \xi q^2 z)\right]Z_1 + 2\beta\mu q^2\left(Z_3 + \frac{1}{2\pi}P\mathrm{e}^{\xi h}\right),$$
$$Y_2(\xi) = \frac{1}{\xi}\left[4\alpha q^2(\lambda+\mu)\xi^2 + m^2q^2 - 2m^2(2\xi^2 - \xi q^2 z) + 2\mu q^2\zeta\right]Z_1 + 2\beta\mu q^2\left(Z_3 + \frac{1}{2\pi}P\mathrm{e}^{\xi h}\right),$$
$$Y_3(\xi) = \mu q^2\left[\beta(\xi+\tau)Z_1 + (\lambda+2\mu)q^2\left(Z_3 + \frac{1}{2\pi}P\mathrm{e}^{\tau h}\right)\right]. \tag{4.26}$$

For $z > h$, the components of displacement vector field and the volume fraction field have the expressions

$$4aq^4u(r,z) = H_1\left\{Y_1(\xi)\mathrm{e}^{-\xi z} - \frac{2}{\tau}\xi^2\Pi(\xi)\mathrm{e}^{-\tau z};\ \ \xi \to r\right\},$$
$$4aq^4w(r,z) = H_0\left\{Y_2(\xi)\mathrm{e}^{-\xi z} - \frac{2}{\tau}\xi^2\Pi(\xi)\mathrm{e}^{-\tau z};\ \ \xi \to r\right\},$$
$$4aq^4\varphi(r,z) = H_0\left\{4q^2\beta\mu Z_1(\xi)\mathrm{e}^{-\xi z} - \frac{2}{\tau}Y_3(\xi)\mathrm{e}^{-\tau z};\ \ \xi \to r\right\}.$$

Let us study now the case of the following concentrated body force

$$f_r = -\frac{F^*}{2\pi r}\delta(r)\delta(z-h), \quad f_z = 0, \quad l = 0, \quad (h > 0), \tag{4.27}$$

where F^* is a given constant. In this case we have

$$\overline{F} = -\frac{F^*}{2\pi}\mathrm{e}^{-ph}, \quad \overline{G} = 0, \quad \overline{L} = 0,$$

and the relation (4.3) imply that

$$N_1 = -\frac{1}{2\pi}F^*\mathrm{e}^{-\xi h}, \quad N_2 = -\frac{1}{2\pi}F^*(\gamma_1 - \xi h)\mathrm{e}^{-\xi h}, \quad N_3 = -\frac{1}{2\pi}F^*k\xi\mathrm{e}^{-\tau h}.$$

It follows from (4.8) that

$$t_{zz}(r,0) = \frac{F^*}{2\pi\rho}\int_0^\infty \frac{1}{\mathrm{d}(\xi)}\left[(q^2h\rho - 2\beta\mu k\xi)\mathrm{e}^{-\xi h} + 2\beta\mu k\xi\mathrm{e}^{-\tau h}\right]\xi^2J_0(\xi r)\,\mathrm{d}\xi,$$
$$t_{zr}(r,0) = -\frac{F^*}{2\pi\rho}\int_0^\infty \frac{1}{\mathrm{d}(\xi)}\left\{\left[q^2\rho(\gamma_2-\gamma_1) + q^2\rho h\xi - 2\beta\mu k\xi\tau\right]\mathrm{e}^{-\xi h} + 2k\beta\mu\xi^2\mathrm{e}^{-\tau h}\right\}\xi J_1(\xi r)\,\mathrm{d}\xi,$$
$$h_z(r,0) = -\frac{F^*k}{2\pi}\int_0^\infty \left[(\gamma_2-\gamma_1)(\mathrm{e}^{-\tau h} - \mathrm{e}^{-\xi h}) + (\tau-\xi)h\mathrm{e}^{-\xi h}\right]\xi^2J_0(\xi r)\,\mathrm{d}\xi. \tag{4.28}$$

We can study the effects inside the medium as in the case of the concentrated extrinsic equilibrated body force.

5. The Neumann problem

In this section we assume that the boundary $z = 0$ is stress-free. Thus we have the boundary conditions

$$t_{zz}(r,0) = 0, \quad t_{zr}(r,0) = 0, \quad h_z(r,0) = 0. \tag{5.1}$$

It follows from (4.7) and (5.1) that

$$(\lambda + 2\mu)W^{(1)} + \lambda\xi U^{(0)} + \beta\Phi^{(0)} = 0, \quad U^{(1)} - \xi W^{(0)} = 0, \quad \Phi^{(1)} = 0. \tag{5.2}$$

In view of (5.2), the relations (3.18) reduce to

$$\begin{aligned} &2\mu\xi(U^{(0)} + W^{(0)}) - \beta\Phi^{(0)} = N_1, \\ &\mu\xi(1 + \gamma_1 + \gamma_2)U^{(0)} + [(\lambda + 2\mu)\xi - \lambda\gamma_1\xi + (\lambda + 2\mu)\gamma_2\xi - \beta\gamma_3]W^{(0)} - (\beta\gamma_2 - \alpha\xi\gamma_3)\Phi^{(0)} = N_2, \\ &2\mu k\tau\xi U^{(0)} + [k(\lambda + 2\mu)\tau^2 - \lambda k\xi^2 - \beta q^2]W^{(0)} + \tau(\alpha q^2 - \beta k)\Phi^{(0)} = N_3, \end{aligned} \tag{5.3}$$

where N_i are given by (4.3).

We introduce the notations

$$\begin{aligned} &M_1 = \mu\xi[\beta(1 + \gamma_1 - \gamma_2) + 2\alpha\xi\gamma_3], \\ &M_2 = \beta[(\lambda + 2\mu)\xi(1 + \gamma_2 - \gamma_1) + 2\mu\xi\gamma_1 - \beta\gamma_3] - 2\mu\xi(\beta\gamma_2 - \alpha\xi\gamma_3), \\ &M_3 = 2\alpha q^2\mu\tau\xi, \\ &M_4 = k\beta\tau[(\lambda + 2\mu)\tau - 2\mu\xi] + q^2(2\mu\xi\alpha\tau - \beta^2) - \beta k\lambda\xi^2, \\ &\Omega = M_1M_4 - M_2M_3, \\ &R_1 = \beta N_2 - (\beta\gamma_2 - \alpha\xi\gamma_3)N_1, \qquad R_2 = \beta N_3 + \tau(\alpha q^2 - \beta k)N_1, \\ &\Psi_1 = R_1M_4 - R_2M_2, \qquad \Psi_2 = R_2M_1 - R_1M_3. \end{aligned} \tag{5.4}$$

From (5.3) we obtain

$$U^{(0)} = \frac{1}{\Omega}\Psi_1, \quad W^{(0)} = \frac{1}{\Omega}\Psi_2, \quad \beta\Phi^{(0)} = 2\mu\xi(U^{(0)} + W^{(0)}) - N_1. \tag{5.5}$$

Thus we can find the components of displacement field and the volume fraction field at the stress-free boundary

$$\begin{aligned} &u(r,0) = H_1\left[\frac{1}{\Omega(\xi)}\Psi_1(\xi); \ \ \xi \to r\right], \\ &w(r,0) = H_0\left[\frac{1}{\Omega(\xi)}\Psi_2(\xi); \ \ \xi \to r\right], \\ &\beta\varphi(r,0) = H_0\left[\frac{2\mu\xi}{\Omega(\xi)}(\Psi_1(\xi) + \Psi_2(\xi)) - N_1(\xi); \ \ \xi \to r\right]. \end{aligned} \tag{5.6}$$

If we use the relations (5.1) and (5.5) then we obtain

$$\frac{\partial w}{\partial z}(r,0) = H_0\left[\frac{1}{\lambda + 2\mu}N_1 - \frac{\xi}{\Omega}\left(\Psi_1 + \frac{2\mu}{\lambda + 2\mu}\Psi_2\right); \ \ \xi \to r\right].$$

It follows from the constitutive equations that the non-vanishing stresses on the boundary $z = 0$ are given by

$$\begin{aligned}
t_{rr}(r,0) &= 2\mu \int_0^\infty \left\{ \frac{\xi}{\Omega}\Psi_1\left[2\xi J_0(\xi r) - \frac{1}{r}J_1(\xi r)\right] + \frac{2\mu}{(\lambda+2\mu)\Omega}\Psi_2\xi^2 J_0(\xi r) - \frac{\xi}{\lambda+2\mu}N_1 J_0(\xi r)\right\} \mathrm{d}\xi,\\
t_{\vartheta\vartheta}(r,0) &= 2\mu \int_0^\infty \left\{ \frac{\xi}{\Omega}\Psi_1\left[\xi J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right] + \frac{2}{\Omega}\xi^2\mu\Psi_2 J_0(\xi r) - \frac{\xi}{\lambda+2\mu}N_1 J_0(\xi r)\right\} \mathrm{d}\xi,\\
h_r(r,0) &= -\frac{\alpha}{\beta}\int_0^\infty \left[\frac{1}{\lambda+2\mu}N_1 - \frac{\xi}{\Omega}\left(\Psi_1 + \frac{2\mu}{\lambda+2\mu}\Psi_2\right)\right]\xi^2 J_1(\xi r)\,\mathrm{d}\xi.
\end{aligned} \tag{5.7}$$

Let us consider now the case of the concentrated extrinsic equilibrated body force defined by (4.11). In this case, from (4.11), (5.3) and (5.4) we obtain

$$\begin{aligned}
&N_1 = 0, \quad N_2 = -\frac{P}{2\pi}\gamma_3 \mathrm{e}^{-\xi h}, \quad N_3 = -\frac{P}{2\pi}q^2 \mathrm{e}^{-\tau h},\\
&\Psi_1 = -\frac{P\beta}{2\pi}(\gamma_3 M_4 \mathrm{e}^{-\xi h} - q^2 M_2 \mathrm{e}^{-\tau h}),\\
&\Psi_2 = -\frac{P\beta}{2\pi}(q^2 M_1 \mathrm{e}^{-\tau h} - \gamma_3 M_3 \mathrm{e}^{-\xi h}).
\end{aligned} \tag{5.8}$$

The functions u, w and φ at the stress-free boundary are given by

$$\begin{aligned}
u(r,0) &= -\frac{P\beta}{2\pi}\int_0^\infty \frac{1}{\Omega(\xi)}(\gamma_3 M_4 \mathrm{e}^{-\xi h} - q^2 M_2 \mathrm{e}^{-\tau h})\xi J_1(\xi r)\,\mathrm{d}\xi,\\
w(r,0) &= -\frac{P\beta}{2\pi}\int_0^\infty \frac{1}{\Omega(\xi)}(q^2 M_1 \mathrm{e}^{-\tau h} - \gamma_3 M_3 \mathrm{e}^{-\xi h})\xi J_0(\xi r)\,\mathrm{d}\xi,\\
\varphi(r,0) &= -\frac{P\mu}{\pi}\int_0^\infty \frac{1}{\Omega(\xi)}[q^2(M_1 - M_2)\mathrm{e}^{-\tau h} + \gamma_3(M_4 - M_3)\mathrm{e}^{-\xi h}]\xi^2 J_0(\xi r)\,\mathrm{d}\xi.
\end{aligned} \tag{5.9}$$

The relations (5.7) reduce to

$$\begin{aligned}
t_{rr}(r,0) = -\frac{P\beta\mu}{\pi}\int_0^\infty \frac{\xi}{\Omega}\Big\{&\left(\gamma_3 M_4 \mathrm{e}^{-\xi h} - q^2 M_2 \mathrm{e}^{-\tau h}\right)\left[2\xi J_0(\xi r) - \frac{1}{r}J_1(\xi r)\right]\\
&+ \frac{2\mu\xi}{\lambda+2\mu}(q^2 M_1 \mathrm{e}^{-\tau h} - \gamma_3 M_3 \mathrm{e}^{-\xi h})J_0(\xi r)\Big\}\,\mathrm{d}\xi,\\
t_{\vartheta\vartheta}(r,0) = -\frac{P\beta\mu}{\pi}\int_0^\infty \frac{\xi}{\Omega}\Big\{&\left(\gamma_3 M_4 \mathrm{e}^{-\xi h} - q^2 M_2 \mathrm{e}^{-\tau h}\right)\left[\xi J_0(\xi r) + \frac{1}{r}J_1(\xi r)\right]\\
&+ 2\mu\xi(q^2 M_1 \mathrm{e}^{-\tau h} - \gamma_3 M_3 \mathrm{e}^{-\xi h})J_0(\xi r)\Big\}\,\mathrm{d}\xi,\\
h_r(r,0) = -\frac{P\alpha}{2\pi}\int_0^\infty \frac{1}{\Omega}\Big[&\gamma_3 M_4 \mathrm{e}^{-\xi h} - q^2 M_2 \mathrm{e}^{-\tau h} + \frac{2\mu}{\lambda+2\mu}(q^2 M_1 \mathrm{e}^{-\tau h} - \gamma_3 M_3 \mathrm{e}^{-\xi h})\Big]\xi^3 J_1(\xi r)\,\mathrm{d}\xi.
\end{aligned}$$

As in the case of Dirichlet problem we can study the effects inside the body.

6. Numerical results

In this section we approximate numerically some of the components of stress tensor, displacement vector and the equilibrated stress. The numerical results are displayed graphically.

If we introduce the notations

$$y^2 = \frac{\zeta}{\alpha}, \quad \kappa^2 = \frac{\alpha}{\zeta} q^2, \quad v = \frac{\lambda}{2(\lambda+\mu)}, \quad \eta = \frac{\beta}{\zeta} y^2, \tag{6.1}$$

then from (4.5) we obtain

$$\rho d = \mu\zeta\omega(\xi), \tag{6.2}$$

where

$$\omega(\xi) = \frac{2(1-v)}{1-2v} y^2\kappa^4 + y^2\kappa^2 + 2(\kappa^2 - 1)\xi(\tau - \xi). \tag{6.3}$$

Clearly, from (3.13) we have

$$\tau = \sqrt{\xi^2 + y^2\kappa^2}.$$

It follows from (4.14) that

$$\begin{aligned}
t_{zz}(r,0) &= -\frac{P}{\pi}\eta\kappa^2 \int_0^\infty \frac{1}{\omega(\xi)} (\mathrm{e}^{-\xi h} - \mathrm{e}^{-\tau h})\xi^2 J_0(\xi r)\,\mathrm{d}\xi, \\
t_{zr}(r,0) &= -\frac{P}{\pi}\eta\kappa^2 \int_0^\infty \frac{1}{\omega(\xi)} (\mathrm{e}^{-\xi h} - \mathrm{e}^{-\tau h})\xi^2 J_1(\xi r)\,\mathrm{d}\xi, \\
h_z(r,0) &= -\frac{P}{\pi} \int_0^\infty \frac{1}{\omega(\xi)} \left\{ y^2\kappa^2 \left[\frac{2(1-v)}{1-2v}\kappa^2 + 1 \right] \mathrm{e}^{-\tau h} - 2(1-\kappa^2)\xi(\tau-\xi)\mathrm{e}^{-\xi h} \right\} \xi J_1(\xi r)\,\mathrm{d}\xi.
\end{aligned} \tag{6.4}$$

On the basis of (2.5) we have $q^2 > 0$, $y^2 > 0$. Clearly,

$$\kappa^2 = 1 - \frac{\beta^2}{\zeta(\lambda + 2\mu)}.$$

Since $\lambda + 2\mu > 0$, we find that the relations (2.5) imply the following restriction

$$\kappa^2 \leqslant 1. \tag{6.5}$$

We present numerical computation for r lying between 0 and 3 assuming that $v = 0.25$, $y^2 = 1$ and $\eta = 1$. We assume that the concentrated load acts at the point $(0, 0, 1)$. We let the coefficient β to vary between zero and infinity and study the variation of stresses. The interval $(0, \infty)$ for α is reduced to the interval (0,1) for κ^2. We consider the values of κ^2 given by 0.25, 0.5 and 0.75.

The integrals which appear in (6.4) are of the form

$$\int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x.$$

The numerical approximations of these integrals have been intensively studied (see, e.g. Krylov, 1962). We use the Gaussian–Laguerre quadrature formulas and the results presented in Gradstein and Rizhik (1971) to approximate them numerically. The variation of the normal force-stress and the equilibrated stress under the action of a concentrated extrinsic equilibrated body force are presented in Figs. 1 and 2.

We now consider Neumann problem in the case of the following concentrated body force

$$f_r = 0, \quad f_z = -\frac{1}{2\pi r} P^* \delta(r)\delta(z-h), \quad \ell = 0, \quad (h > 0), \tag{6.6}$$

where P^* is a given constant. From (4.3) we obtain

$$N_1 = \mathscr{M}\mathrm{e}^{-\xi h}, \quad N_2 = \mathscr{M}(\gamma_2 - \xi h)\mathrm{e}^{-\xi h}, \quad N_3 = k\tau\mathscr{M}^{-\tau h}, \tag{6.7}$$

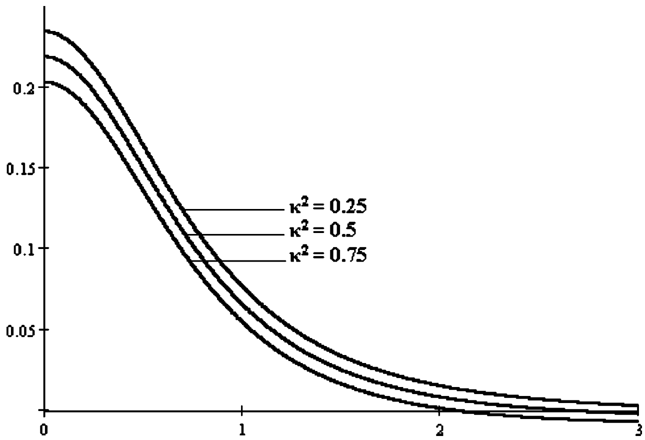

Fig. 1. Variation of the normal force-stress $(-2\pi/P)t_{zz}(r,0)$ with r and κ^2 under the action of a concentrated extrinsic equilibrated body force.

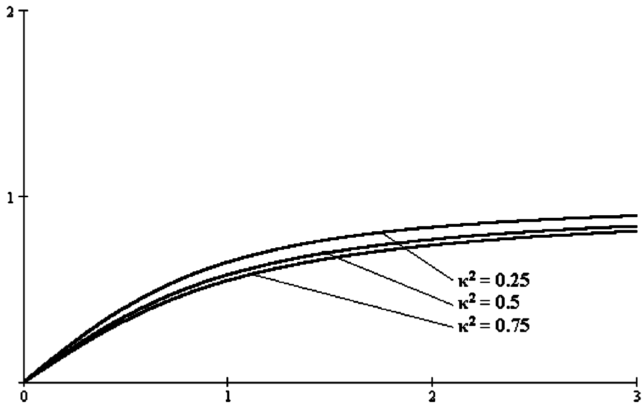

Fig. 2. Variation of the normal equilibrated stress $(-2\pi/P)h_z(r,0)$ with r and κ^2 under the action of a concentrated extrinsic equilibrated body force.

where

$$\mathscr{M} = -\frac{1}{2\pi}P^*.$$

In what follows we assume that $v = 0.25$, $y^2 = 1$, $\eta = 1$ and $h = 1$. From (5.4) we find that

$$\begin{aligned} &\rho M_1 = 2\beta^2\mu^2\xi(2\xi^2 - 3\kappa^2), \qquad \rho M_2 = 2\beta^2\mu^2\xi(2\xi^2 - 1), \\ &M_3 = 2\beta\mu\kappa^2\tau\xi, \qquad M_4 = \tfrac{2}{3}\beta\xi(\beta\xi - \beta\tau + 3\kappa^2\mu\tau), \end{aligned} \tag{6.8}$$

where

$$\Omega_1 = (2\xi^2 - 3\kappa^2)[\beta(\xi - \tau) + 3\kappa^2\mu\tau] - 3\kappa^2\mu\tau(2\xi^2 - 1). \tag{6.9}$$

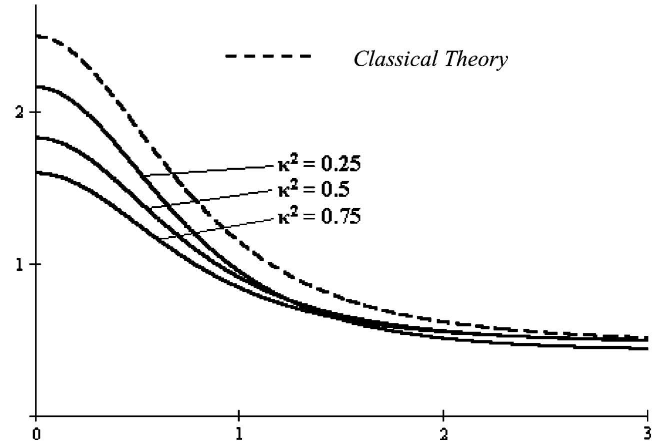

Fig. 3. Variation of the normal displacement $(-4\pi\mu/P^*)w(r,0)$ with r and κ^2 on the stress-free boundary of the half-space under the action of a concentrated body force.

It follows from (5.4) and (6.7) that

$$\begin{aligned}\rho R_1 &= \beta^2\mu\xi\mathscr{M}e^{-\xi}(3\kappa^2+2\xi-1),\\ R_2 &= \tau\beta\mathscr{M}[(1-\kappa^2)e^{-\tau}+(2\kappa^2-1)e^{-\xi}],\\ \rho\Psi_2 &= 2\beta^3\mu^2\tau\xi\mathscr{M}\{(2\xi^2-3\kappa^2)\left[(1-\kappa^2)e^{-\tau}+(2\kappa^2-1)e^{-\xi}\right]-\kappa^2\xi(3\kappa^2+2\xi-1)e^{-\xi}\}.\end{aligned} \tag{6.10}$$

Thus, from (5.6) we obtain

$$(-4\pi\mu/P^*)w(r,0)=\int_0^\infty\{(1-\kappa^2)(2\xi^2-3\kappa^2)e^{-\tau}+\left[(3+\xi+2\xi^2)\kappa^2-3(2+\xi)\kappa^4-2\xi^2\right]e^{-\xi}\}\tau V(\xi)J_0(\xi r)\,dr, \tag{6.11}$$

where

$$V^{-1}=(2\xi^2-3\kappa^2)(1-\kappa^2)(\xi-\tau)+\kappa^2\tau(1-3\kappa^2). \tag{6.12}$$

In Fig. 3 we show the variation of the normal displacement on the stress-free boundary of the half-space which is subjected to the action of a concentrated body force applied at the point $(0,0,1)$ of the half-space.

We conclude that for small values of r there is a difference between the solution of the classical elastostatics and the solution of the problem in the context of the theory of elastic materials with voids. We note that this difference is larger with the increasing values of the constant β. In a similar way we can study the behavior of the radial displacement and the volume fraction function.

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